Chapter 3.2 - 3-Connected Graphs

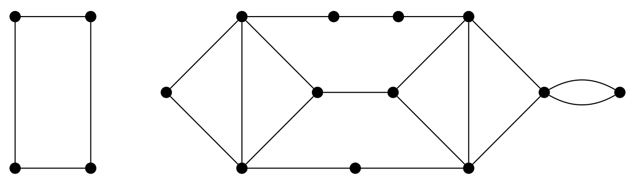
Recall: A graph G is called k-**connected** (for $k \in \mathbb{N}$) if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. Largest k such that G is k-connected is called **connectivity** of G, denoted $\kappa(G)$.

Notice $\kappa(K_n) = n - 1$.

Goals:
Describe construction(s) of all 3-connected graphs. The process will be if G is not K_4, then there is an edge that can be deleted (and suppress vertices of degree 2) or contracted.

Suppressing a vertex v of degree 2 in a multigraph G means deleting v and adding a new edge between the two neighbors of v. If the neighbors of v are the same vertex $w \neq v$, we add a loop at w. If v is incident with a loop, we simply delete v.

1: Suppress all vertices of degree 2, one by one, until there are no vertices of degree 2 in the following multigraph.

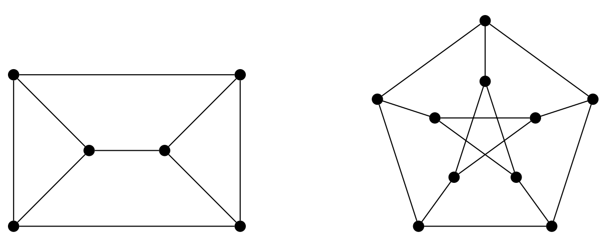


Subdivision of a graph X is a graph G obtained from X by replacing each edge of X by a path. Or one can thing that each edge of X can be subdivided (many times).

TX are all subdivisions of X. Sometimes we use TX as a particular instance.

X is a **topological minor** of G if G contains TX as a subgraph.

2: Find a topological minor TK_4 in the following graphs.



$G \dot{-} e$ denotes the multigraph obtained from $G - e$ by suppressing end-vertices of e of degree 2 in $G - e$.

Lemma Let G be a graph and e be its edge. If $G \dot{-} e$ is a 3-connected graph, then G is also 3-connected.

3: Prove the lemma.

Solution: Call vertices in $G \dot{-} e$ old and possible new endpoints y_1, y_2. If there are no new vertices, G is clearly 3-connected. Let y_1 be a new vertex. Note is has 3 distinct neighbors. So it cannot be separated from the rest by a 2-cut. If y_1 is in a 2-cut, it can be replaces by one of its neighbor to get a 2-cut in $G \dot{-} e$. Fun part if y_1, y_2 are new vertices and they would create a cut, there would be a 2-edge cut in $G \dot{-} e$.

4: Show that if $G \dot{-} e$ is a 3-connected multigraph, then G is not necessarily 3-connected.

Solution: Add edge e into a bigon.

5: Show that if G is 3-connected then it contains TK_4.

Solution: Take a shortest cycle C in G. Then take a path C-path P. Notice P has an internal vertex. Let u, v be the common vertices of P and C. Since G is 3-connected, there is C-P-path Q. Now find TK_4 in $P \cup C \cup Q$.

Lemma If $G \neq K_4$ is a 3-connected graph, then it contains an edge e such that $G \dot{-} e$ is also 3-connected.

Proof Find 3-connected J such that $J \neq G$ and G contains TJ. (Why such J exists?). Pick one with where $||J|| = |E(G)|$ maximized. Then pick $H = TJ$ that is a subgraph of G, where $||H||$ is also maximized. Goal is to find e such that $G \dot{-} e \cong J$.

6: Show that $H \neq G$.

Solution: The difference between H and J are some vertices of degree 2. If they are there, H is not connected. If they are not there, $H = J \neq G$.

Since $H \neq G$, there exists an H-path $P = u, \ldots, v$.

7: Show that it is possible to pick P such that u and v are NOT on the same subdivided edge of J.
Hint: First show $H = J$. Then show that P would contradict maximality of H.

Solution: If $H \neq J$, then there is a vertex z inside a path $xy \in E(J)$. Now there exists in $G - \{x, y\}$ a z-J, which works for the choice of P.

Now $H \neq J$. If there is an x-y path P for some $xy \in E(J)$, we can replace xy in H by P and obtain a contradiction. We used G has no parallel edges, so P has an inner vertex.

Now consider $J' = H \cup P$ and suppress all vertices of degree 2. The path P turns into an edge e after the suppression.

8: Notice that $J' \dot{-} e = J$ and finish the proof of the lemma.

Solution: J' is 3-connected because J is. Because J was largest, $J' = G$ and we are done.

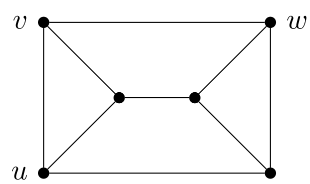
Theorem (Tutte 1966)
A graph G is 3-connected if and only if there exists a sequence $G_0, \dots, G_n$ of graphs such that
(i) $G_0 = K_4$ and $G_n = G$;
(ii) G_{i+1} has an edge e such that $G_i = G_{i+1} \dot{-} e$, for every $1 \leqslant i < n$. Moreover, the graphs in any such sequence are all 3-connected.

9: Prove Tutte's theorem.

Solution: Just use the previous lemma iteratively.

G/e is **contraction** of an edge. If $e = uv$, then G/e is obtained from G by removing u, v and adding new vertex x incident to $(N_G(u) \cup N_G(v)) \setminus \{u, v\}$. Note there is also version, where G/e may be a multigraph.

10: Find graphs G/vw and G/uv.



Lemma Every 3-connected graph $G \neq K_4$ has an edge e such that G/e is also 3-connected.

Proof Suppose for contradiction there is no such edge.

11: Show that for each edge xy exists z such that xyz is a cut in G.

Solution: Consider G/xy. By our assumptions, G/xy is not 3-connected but it is 2-connected. If G/xy not 2-connected, G not 3-connected. Clearly the new vertex, call it w must be in the 2-cut. And then there is another vertex z to finish the 2-cut.

Now pick edge xy and z forming a cut in G. Let $S = \{x, y, z\}$.

12: Let C be a component in $G - S$. Show that each vertex in S has a neighbor in C.

Solution: If not, we have a 2-cut, contradiction with 3-connectivity of G.

We pick a particular S such that a component in $G - S$ is as small as possible, call this component C. By the previous observation, z has a neighbor v in C.

No there exists w such that $\{z, v, w\}$ is a 3-cut in G.

13: Show that $G - \{z, v, w\}$ has a smaller component than C, contradicting the minimality of $|C|$.
Hint: Consider neighbors of v.

Solution:

Theorem (Tutte 1961)
A graph G is 3-connected if and only if there exists a sequence $G_0, \ldots, G_n$ of graphs with the following two properties:
(i) $G_0 = K_4$ and $G_n = G$;
(ii) G_{i+1} has an edge xy such that $d(x), d(y) \geqslant 3$ and $G_i = G_{i+1}/xy$, for every $i < n$.
Moreover, the graphs in any such sequence are all 3-connected.

14: Prove the Theorem.

Solution: